Euclidean Jordan Algebras and Kepler Problem

Guowu Meng

Department of Mathematics
Hong Kong University of Science and Technology

Think deeply of simple things — Arnold Ross

We have learned from the last lecture that there is a family of God-given classical dynamic models (indexed by a real parameter μ) with

- configuration space $\mathbb{R}^3_*$,
- conserved angular momentum $\mathbf{L}$,
- and an additional conserved vector $\mathbf{A}$.

Moreover, the orbits of these models are either linear or conic. These models are completely integrable in the sense that

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We also learned that the orbits of these models have very attractive descriptions on the future light cone

$$\Lambda_+ := \{x \in \mathbb{R}^{1,3} | x^2 = 0, x_0 > 0\}$$

in the Minkowski space $\mathbb{R}^{1,3} := (\mathbb{R} \oplus \mathbb{R}^3, \text{Lorentz inner product})$ spanned by our ordinary three spatial dimensions and a new mysterious temporal dimension. Is this a coincidence? More precisely, one may ask this

Question: Can Kepler problem and its magnetized versions be naturally formulated on that future light cone Λ_+?

Answer: Yes, provided that we can employ the more refined Jordan algebra structure behind the Lorentz structure on that Minkowski space $\mathbb{R}^{1,3}$. Since

$$\begin{array}{rcl} \mathbb{R}^3_* & \to & \Lambda_+ \\ \mathbf{r} & \mapsto & (r, \mathbf{r}) \end{array}$$

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The Jordan algebra structure on $\mathbb{R}^{1,3}$

Write $\mathbf{x} = (x_1, x_2, x_3)$, then $x = (x_0, x_1, x_2, x_3)$. Write X for $x_0 I + \mathbf{x} \cdot \vec{\sigma}$, i.e.

$$X = \begin{bmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{bmatrix}.$$

Let $\mathrm{H}_2(\mathbb{C})$ is the set of all complex hermitian matrices of order two. Note that $\det X = x^2$.

- The map $x \mapsto X$ is an isometry between $\mathbb{R}^{1,3}$ and $(\mathrm{H}_2(\mathbb{C}), \det)$.
- Under the symmetrized matrix multiplication:

$$X \circ Y := \frac{1}{2}(XY + YX),$$

  $\mathrm{H}_2(\mathbb{C})$ becomes a *real commutative algebra with unit*.
- This algebra is formally real in the following sense: for A, B in $\mathrm{H}_2(\mathbb{C})$, $A^2 + B^2 = 0 \implies A = B = 0$.

Indeed, for any column vector $\vec{x}$ in $\mathbb{C}^2$, let $\vec{x}^\dagger$ be its hermitian conjugate. Then $0 = \vec{x}^\dagger(A^2 + B^2)\vec{x} = ||A\vec{x}||^2 + ||B\vec{x}||^2$, so $A\vec{x} = B\vec{x} = \vec{0}$, so $A = B = 0$.

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Weak associativity

The symmetrized matrix multiplication is not associative. But it is **weakly associative** in the following sense: for X, Y in $\mathrm{H}_2(\mathbb{C})$, we have

$$(X \circ Y) \circ X^2 = X \circ (Y \circ X^2).$$

Here $X^2 = X \circ X = XX$.

Proof.

$$\begin{aligned} LHS &= \frac{1}{2}(XY + YX) \circ X^2 = \frac{1}{4}[(XY + YX)X^2 + X^2(XY + YX)] \\ &= \frac{1}{4}[XYX^2 + YXX^2 + X^2XY + X^2YX] \\ &= \frac{1}{4}[XYX^2 + YX^2X + XX^2Y + X^2YX] \\ &= \frac{1}{4}[(YX^2 + X^2Y)X + X(YX^2 + X^2Y)] = X \circ \frac{1}{2}(YX^2 + X^2Y) \\ &= RHS \end{aligned}$$

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The euclidean structure on $\mathrm{H}_2(\mathbb{C})$

For any $u \in \mathrm{H}_2(\mathbb{C})$, we let L_u be the endomorphism on $\mathrm{H}_2(\mathbb{C})$ defined by $v \mapsto u \circ v$. Let $\langle\, ,\, \rangle\colon \mathrm{H}_2(\mathbb{C}) \times \mathrm{H}_2(\mathbb{C}) \to \mathbb{R}$ be defined as follows:

$$\langle u, v\rangle := \frac{1}{2}\mathrm{tr}\,(u \circ v) = \frac{1}{2}\mathrm{tr}\,(uv) = \frac{1}{4}\mathrm{tr}\, L_{u\circ v}.$$

- $\langle\, ,\, \rangle$ is an inner product on $\mathrm{H}_2(\mathbb{C})$ such that
$$\sigma_0 := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \sigma_1 := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_2 := \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \sigma_3 := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
form an orthonormal basis. Note that $\mathrm{tr}\,\sigma_0 = 2$ and $\mathrm{tr}\,\sigma_i = 0$.
- The multiplication law for the real commutative algebra $\mathrm{H}_2(\mathbb{C})$ with unit is given by
$$\sigma_i \circ \sigma_j = \delta_{ij}\sigma_0, \quad \sigma_0 \text{ is the unit } e.$$
- L_u is self-adjoint with respect to $\langle\, ,\, \rangle$, i.e., $\langle v, u\circ w\rangle = \langle u \circ v, w\rangle$ for any $v, w \in \mathrm{H}_2(\mathbb{C})$. Indeed,
$$LHS = \frac{1}{2}\mathrm{tr}\,(v(uw+wu)) = \frac{1}{2}\mathrm{tr}\,(vuw+vwu) = \frac{1}{2}\mathrm{tr}\,((uv+vu)w) = RHS$$

Relevance to Kepler problem

- The future light cone Λ_+ = the set of rank one, semi-positive elements in $\mathrm{H}_2(\mathbb{C})$. Indeed, if the rank of X is less than two, then $\det X = 0$, also, if $X \neq 0$ is semi-positive, then $\operatorname{tr} X > 0$. So $x^2 = 0$ and $x_0 > 0$.
- For the Kepler problem, the potential term is

$$-\frac{1}{\langle e, x\rangle}.$$

- The Kinetic term for the the Kepler problem (or rather the Riemannian metric on Λ_+), angular momentum, and Lenz vector can all be naturally expressed in terms of Jordan algebra structure as well.

For details, we have to study Formally Real Jordan algebras.

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Formally real Jordan algebras [J.Faraut and A.Koranyi, *Analysis on Symmetric Cones*]

Jordan algebras are the unfavored cousins of Lie algebras, and Formally real Jordan algebras are the unfavored cousins of compact real Lie algebras. Having $\mathrm{H}_2(\mathbb{C})$ in mind, we have

Definition (P. Jordan, 1933)

A finite dimensional **formally real Jordan algebra** is a finite dimensional real algebra V with unit e such that, for any elements a, b in V, we have

1) $ab = ba$ (symmetry),
2) $a(ba^2) = (ab)a^2$ (weakly associative),
3) $a^2 + b^2 = 0 \implies a = b = 0$ (formally real).

The simplest example is $\mathbb{R}$, the other example is $\mathrm{H}_2(\mathbb{C})$. We use L_a: $V \to V$ to denote the multiplication by a. Then 2) says that $[L_a, L_{a^2}] = 0$ (Jordan Identity) and 3) can be replaced by

3´) The "Killing form" $\langle a \mid b\rangle = \frac{1}{\dim V}\mathrm{tr}\, L_{ab}$ is positive definite. Note that $\langle b \mid ac\rangle = \langle ab \mid c\rangle$. Formally real Jordan algebras are also called Euclidean Jordan algebras.

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The classification theorem

Theorem (Jordan, von Neumann and Wigner, 1934)

Euclidean Jordan algebras are semi-simple, and the simple ones consist of four infinity families and one exceptional:

$\mathbb{R}$.

$\Gamma(n) := \mathbb{R} \oplus \mathbb{R}^n$, $n \geq 2$.

$\mathrm{H}_n(\mathbb{R})$, $n \geq 3$.

$\mathrm{H}_n(\mathbb{C})$, $n \geq 3$.

$\mathrm{H}_n(\mathbb{H})$, $n \geq 3$.

$\mathrm{H}_3(\mathbb{O})$.

Remark.

- $\Gamma(0) \cong \mathbb{R}$, $\Gamma(1) \cong \mathbb{R} \oplus \mathbb{R}$, $\Gamma(2) \cong \mathrm{H}_2(\mathbb{R})$, $\Gamma(3) \cong \mathrm{H}_2(\mathbb{C})$, $\Gamma(5) \cong \mathrm{H}_2(\mathbb{H})$, $\Gamma(9) \cong \mathrm{H}_2(\mathbb{O})$.
- Each but the exceptional one is associated with an associative algebra.
- $\mathbb{R}$, $\Gamma(3)$, and $\mathrm{H}_3(\mathbb{O})$ are somewhat special.

The structure algebra

For a, b in V, we let

$$S_{ab} := [L_a, L_b] + L_{ab}, \quad \{abc\} := S_{ab}(c)$$

and $\mathfrak{str}$ be the span of $\{S_{ab} \mid a, b \in V\}$ over $\mathbb{R}$. Since

$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}},$$

$\mathfrak{str}$ becomes a real Lie algebra — the **structure algebra** of V. For example, (1) $\mathfrak{str} \cong \mathbb{R}$ for $V = \mathbb{R}$, (2) $\mathfrak{str} \cong \mathfrak{so}(1,3) \oplus \mathbb{R}$ for $V = \Gamma(3)$.

This Lie algebra is not simple, actually not even semi-simple, because it has a non-trivial central element: S_{ee}. Note that $L_u = S_{ue}$.

The good news is that this algebra can be extended to a simple real Lie algebra provided that V is a simple Euclidean Jordan algebra. From hereon V is assumed to be a simple Euclidean Jordan algebra .

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$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}},$$

$\mathfrak{str}$ becomes a real Lie algebra — the **structure algebra** of V. For example, (1) $\mathfrak{str} \cong \mathbb{R}$ for $V = \mathbb{R}$, (2) $\mathfrak{str} \cong \mathfrak{so}(1,3) \oplus \mathbb{R}$ for $V = \Gamma(3)$.

This Lie algebra is not simple, actually not even semi-simple, because it has a non-trivial central element: S_{ee}. Note that $L_u = S_{ue}$.

The good news is that this algebra can be extended to a simple real Lie algebra provided that V is a simple Euclidean Jordan algebra. From hereon V is assumed to be a simple Euclidean Jordan algebra .

The conformal algebra

Write $z \in V$ as X_z and $\langle w \mid \ \rangle \in V^*$ as Y_w.

Definition (J. Tits, M. Koecher, I.L. Kantor, 1960's)

The **conformal algebra** $\mathfrak{co}$ is a Lie algebra whose underlying real vector space is $V \oplus \mathfrak{str} \oplus V^*$, and the commutation relations are

$$\boxed{\begin{array}{c} [X_u, X_v] = 0, \quad [Y_u, Y_v] = 0, \quad [X_u, Y_v] = -2S_{uv}, \\[2ex] [S_{uv}, X_z] = X_{\{uvz\}}, \quad [S_{uv}, Y_z] = -Y_{\{vuz\}}, \\[2ex] [S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}} \end{array}} \tag{1}$$

for u, v, z, w in V.

When $V = \Gamma(3)$, $\mathfrak{str} = \mathfrak{so}(3,1) \oplus \mathbb{R}$, $\mathfrak{co} = \mathfrak{so}(4,2)$. When $V = \mathbb{R}$, $\mathfrak{str} = \mathbb{R}$, $\mathfrak{co} = \mathfrak{sl}(2, \mathbb{R})$. In general, $\mathfrak{co}$ is the Lie algebra of the bi holomorphic automorphism group of the complex domain $V \oplus \mathrm{i} V_+ \subset V \otimes_{\mathbb{R}} \mathbb{C}$.

After spending so much effort on the basic facts on Euclidean Jordan algebras, an impatient audience may ask this

Question: How could the Euclidean Jordan algebra $\mathrm{H}_2(\mathbb{C})$ (or $\Gamma(3)$) be relevant to the Kepler problem?

Well, the answer will become clear after we review the Lenz algebra for the Kepler problem.

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Well, the answer will become clear after we review the Lenz algebra for the Kepler problem.

Lenz algebra for the Kepler problem

The phase space for the Kepler problem, i.e., $T^*\mathbb{R}^3_*$, is a Poisson manifold. In terms of the standard canonical coordinates $x^1, x^2, x^3, p_1, p_2, p_3$, the Poisson structure can be described by the following basic Poisson bracket relations:

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0.$$

Recall that the Hamiltonian, angular momentum, and Lenz vector are

$$\mathrm{H} = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r}, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{A} = \mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}$$

respectively. In terms of Poisson bracket, the fact that $\mathbf{L}$ and $\mathbf{A}$ are constants of motion can be restated as

$$\{\mathbf{L}, \mathrm{H}\} = 0, \quad \{\mathbf{A}, \mathrm{H}\} = 0.$$

To show that, we first note that L_i (the i-th component of $\mathbf{L}$) is the infinitesimal generator of the rotation about the i-th axis. For example, since $L_3 = x^1 p_2 - x^2 p_1$, we have

$$\{L_3, x^1\} = -x^2\{p_1, x^1\} = x^2, \{L_3, x^2\} = -x^1, \{L_3, x^3\} = 0.$$

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$$\{L_3, x^1\} = -x^2\{p_1, x^1\} = x^2, \{L_3, x^2\} = -x^1, \{L_3, x^3\} = 0.$$

Similarly, we have

$$\{L_3, p_1\} = p_2, \quad \{L_3, p_2\} = -p_1, \quad \{L_3, p_3\} = 0.$$

Then, it is clear that $\{\mathbf{L}, \mathrm{H}\} = 0$; moreover,

$$\begin{aligned}
\{\mathbf{A}, \mathrm{H}\} &= \mathbf{L} \times \{\mathbf{p}, \mathrm{H}\} + \{\frac{\mathbf{r}}{r}, \mathrm{H}\} \\
&= \mathbf{L} \times \{\mathbf{p}, -\frac{1}{r}\} + \{\frac{\mathbf{r}}{r}, \frac{1}{2}\mathbf{p}^2\} \\
&= \mathbf{L} \times \nabla\frac{1}{r} + \sum_i \{\frac{\mathbf{r}}{r}, p_i\} p_i \\
&= -\mathbf{L} \times \frac{\mathbf{r}}{r^3} + \sum_i p_i \partial_{x^i} \frac{\mathbf{r}}{r} \\
&= -(\mathbf{r} \times \mathbf{p}) \times \frac{\mathbf{r}}{r^3} + \frac{\mathbf{p}}{r} - \frac{\mathbf{r} \cdot \mathbf{p}}{r^3}\mathbf{r} \\
&= \mathbf{0}.
\end{aligned}$$

In fact, it is fairly routine to verify this

Theorem

Let L_i (resp. A_i) be the i-th component of **L** *(resp.* **A***). Then*

$$\boxed{\begin{array}{rcl} \{L_i, \mathrm{H}\} & = & 0\,, \\ \{A_i, \mathrm{H}\} & = & 0\,, \\ \{L_i, L_j\} & = & \epsilon_{ijk} L_k\,, \\ \{L_i, A_j\} & = & \epsilon_{ijk} A_k\,, \\ \{A_i, A_j\} & = & -2\mathrm{H}\epsilon_{ijk} L_k\,. \end{array}} \tag{2}$$

Here $\epsilon_{ijk} = 1$ (resp. -1) if ijk is an even (resp. odd) permutation of 123, and equals to 0 otherwise. A summation over the repeated index k is assumed. So we have $\{L_1, L_2\} = L_3$, $\{L_2, A_3\} = A_1$, and so on.

The real associated algebra with generators $H, L_1, L_2, L_3, A_1, A_2, A_3$ and relations in Eq. (2) is called the Lenz algebra.

With this in mind, we are now ready to introduce the Universal Kepler Problem.

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Universal Kepler problem

[Based on [G. Meng, "The Universal Kepler Problems", JGSP 36 (2014) 47-57]] Let $\mathcal{TKK}$ be the complexified universal enveloping algebra for the conformal algebra, but with Y_e being formally inverted.

Definition

The **universal angular momentum** is

$$\begin{array}{rcl} L : V \times V & \to & \mathcal{TKK} \\ (u, v) & \mapsto & L_{u,v} := [L_u, L_v] \end{array} \tag{3}$$

The **universal Hamiltonian** is

$$H := \frac{1}{2} Y_e^{-1} X_e - (\mathrm{i} Y_e)^{-1} \tag{4}$$

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Universal Lenz algebra

Via the commutation relation for the conformal algebra, one can verify

Theorem

For u, v, z and w in V,

$$\boxed{\begin{array}{lcl} [L_{u,v}, H] & = & 0\ , \\ [A_u, H] & = & 0\ , \\ [L_{u,v}, L_{z,w}] & = & L_{L_{u,v}z,w} + L_{z,L_{u,v}w}\ , \\ [L_{u,v}, A_z] & = & A_{L_{u,v}z}\ , \\ [A_u, A_v] & = & -2HL_{u,v}\ . \end{array}} \tag{6}$$

Proof. Since $S_{uv} = L_{u,v} + L_{uv}$, part of the commutation relations for the conformal algebra can be rewritten as

$$[L_{u,v}, X_z] = X_{L_{u,v}z}, \quad [L_{u,v}, Y_z] = Y_{L_{u,v}z},$$

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Then we have $[L_{u,v}, X_e] = [L_{u,v}, Y_e] = 0$, so $[L_{u,v}, H] = 0$.

Also, we have

$$\begin{aligned}[L_{u,v}, A_z] &= \mathrm{i}\, Y_e^{-1}[L_{u,v}, [L_z, Y_e^2 H]] \\ &= \mathrm{i}\, Y_e^{-1}\left([[L_{u,v}, L_z], Y_e^2 H] + [L_z, [L_{u,v}, Y_e^2 H]]\right) \\ &= \mathrm{i}\, Y_e^{-1}[L_{L_{u,v}z}, Y_e^2 H] \\ &= A_{L_{u,v}z}.\end{aligned}$$

The rest of the proof is skipped. □

A concrete realization of the conformal algebra

$\Downarrow$

a concrete model of the Kepler type

To be more precise, we have

A suitable operator realization $\Longrightarrow$ a quantum model.

A suitable Poisson realization $\Longrightarrow$ a classical model.

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Poisson realizations

In a Poisson realization of the TKK algebra, S_{uv}, X_z, Y_w are respectively represented as real functions $\mathcal{S}_{uv}$, $\mathcal{X}_z$, $\mathcal{Y}_w$ on a Poisson manifold so that the commutation relations are represented by the Poisson bracket relations: for u, v, z, w in V, we have

$$\boxed{\begin{gathered}\{\mathcal{X}_u, \mathcal{X}_v\} = 0, \quad \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, \quad \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, \quad \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}.\end{gathered}} \tag{7}$$

Then, H, A_u and $L_{u,v}$ can be realized as real functions

$$\mathcal{H} = \frac{\frac{1}{2}\mathcal{X}_e - 1}{\mathcal{Y}_e}, \quad \mathcal{A}_u := \frac{\{\mathcal{L}_u, \mathcal{Y}_e^2\mathcal{H}\}}{\mathcal{Y}_e}, \quad \mathcal{L}_{u,v} := \{\mathcal{L}_u, \mathcal{L}_v\} \tag{8}$$

respectively. Note that

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Poisson Realization on TV

Via the canonical inner product on V, $TV \cong T^*V$. So TV becomes a symplectic space. Denote an element of $TV = V \times V$ by (x, π) and fix an orthonormal basis $\{e_\alpha\}$ for V so that we can write $x = x^\alpha e_\alpha$ and $\pi = \pi^\alpha e_\alpha$. Then the basic Poisson bracket relations on TV are

$$\{x^\alpha, \pi^\beta\} = \delta^{\alpha\beta}, \quad \{x^\alpha, x^\beta\} = 0, \quad \{\pi^\alpha, \pi^\beta\} = 0.$$

In coordinate free form, we have

$$\{\langle x \mid u\rangle, \langle \pi \mid v\rangle\} = \langle u \mid v\rangle, \quad \{\langle x \mid u\rangle, \langle x \mid v\rangle\} = \{\langle \pi \mid u\rangle, \langle \pi \mid v\rangle\} = 0.$$

One can check that real functions

$$\mathcal{S}_{uv} := \langle S_{uv}(x) \mid \pi\rangle, \quad \mathcal{X}_u := \langle x \mid \{\pi u \pi\}\rangle, \quad \mathcal{Y}_v := \langle x \mid v\rangle \tag{10}$$

yield a Poisson realization on TV of S_{uv}, X_z, Y_w respectively.
Proof. It is clear that $\{\mathcal{Y}_u, \mathcal{Y}_v\} = 0$.

$$\begin{aligned}
\{\mathcal{X}_u, \mathcal{Y}_v\} &= \{\langle x \mid \{\pi u \pi\}\rangle, \langle x \mid v\rangle\} \\
&= -2\langle x \mid \{v u \pi\}\rangle = -2\langle S_{uv}(x) \mid \pi\rangle \\
&= -2\mathcal{S}_{uv}.
\end{aligned}$$

Poisson Realization on TV

Via the canonical inner product on V, $TV \cong T^*V$. So TV becomes a symplectic space. Denote an element of $TV = V \times V$ by (x, π) and fix an orthonormal basis $\{e_\alpha\}$ for V so that we can write $x = x^\alpha e_\alpha$ and $\pi = \pi^\alpha e_\alpha$. Then the basic Poisson bracket relations on TV are

$$\{x^\alpha, \pi^\beta\} = \delta^{\alpha\beta}, \quad \{x^\alpha, x^\beta\} = 0, \quad \{\pi^\alpha, \pi^\beta\} = 0.$$

In coordinate free form, we have

$$\{\langle x \mid u\rangle, \langle \pi \mid v\rangle\} = \langle u \mid v\rangle, \quad \{\langle x \mid u\rangle, \langle x \mid v\rangle\} = \{\langle \pi \mid u\rangle, \langle \pi \mid v\rangle\} = 0.$$

One can check that real functions

$$\mathcal{S}_{uv} := \langle S_{uv}(x) \mid \pi\rangle, \quad \mathcal{X}_u := \langle x \mid \{\pi u \pi\}\rangle, \quad \mathcal{Y}_v := \langle x \mid v\rangle \tag{10}$$

yield a Poisson realization on TV of S_{uv}, X_z, Y_w respectively.

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Poisson Realization on TV

Via the canonical inner product on V, $TV \cong T^*V$. So TV becomes a symplectic space. Denote an element of $TV = V \times V$ by (x, π) and fix an orthonormal basis $\{e_\alpha\}$ for V so that we can write $x = x^\alpha e_\alpha$ and $\pi = \pi^\alpha e_\alpha$. Then the basic Poisson bracket relations on TV are

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&= \langle [S_{uv}, S_{zw}](x) \mid \pi\rangle = \langle (S_{\{uvz\}w} - S_{z\{vuw\}})(x) \mid \pi\rangle \\
&= \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}.
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The rest of the proof is skipped. □

However, this is not a suitable Poisson realization because neither $\mathcal{X}_e$ nor $\mathcal{Y}_e$ is positive on TV.

To salvage this Poisson realization, we restrict the Poisson realization to certain sub-symplectic manifolds of TV, for example, $T\mathcal{C}_r$ where $\mathcal{C}_r$ is the set of rank r semi-positive elements of V, with r being a positive integer less than or equal to the rank of V. Indeed, restricting $\mathcal{H}$ to $T\mathcal{C}_r$ yields an integrable model of Kepler type, which is the Kepler problem when $V = \Gamma(3)$ and $r = 1$.

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